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GEOMETRY OF OPTIMAL SHOCK-WAVE SYSTEMS

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The notion of optimal shock-wave systems consisting of several plane oblique shocks and a closing normal shock was introduced at the end of the 1940s by Petrov and Uskov [1] and also by Osvatich (see the bibliography in [2]). The oblique-shock intensities at which maximum recovery factors of static and total pressures are reached in the system have been determined numerically [1] and analytically [2].

A detailed analysis of optimal systems has been performed previously [3]. In the present paper, as an extension of [3], we study the geometry of bodies such that the flow past them leads to the formation of optimal shock-wave systems. We give rigorous analytic solutions that determine the flow deflection angle in optimal systems in which there are extrema not only of the pressure recovery factors but also of the values of velocity head and density. Emphasis is on the analysis of optimal multishock systems at large Mach numbers.

The resulting solutions are of both theoretical and applied significance and can be used in the gas-dynamic design of supersonic air intake, jet devices, and other technical objects.

1. We consider a plane supersonic flow of a perfect inviscid gas which passes through a system S_n of n waves (shock or isentropic). In the system $F = \{p, \rho, T, \rho v^2, p_0, \rho_0, T_0\}$, the set of gas-dynamic variables S_n that characterizes the undisturbed flow is transformed to the corresponding set F_n of gas-dynamic variables past S_n.

Often, the state of the flow past the system is [1–5] characterized by the recovery factors $K_n^{(f)}$ of gas-dynamic variables, which are the ratio of the elements of the set to the corresponding deceleration parameters of the undisturbed flow.

It has been shown [3] that, for a given adiabatic exponent F_n and a given Mach number γ of the undisturbed flow, any of the recovery factors can be expressed in terms of the ratio of the static pressures $J_s \equiv p_n/p$ ahead of and behind the system, which is frequently called the intensity of the system.

It is easy to see that the quantity J_s is equal to the product of the intensities $J_k = p_k/p_{k-1}$ of all the waves that enter the system. Hence, for any $f \in F$, the quantity $K_n^{(f)}$ is a function of n variables — the wave intensities J_k.

The analysis performed in [1–3] showed that some of the functions $K_n^{(f)}$ are nonmonotonic and reach extremum at certain values $J_k^{(f)}$. Shock-wave systems with the intensities $J_k^{(f)}$ are called optimal systems for the variables f.

The flow properties past an optimal shock-wave system depend greatly on the intensity of the closing shock σ. Usually, this shock is especially distinguished, and an optimal system with a closing shock is denoted by $S_{n,\sigma}$.

In some systems, the type of closing shock can be chosen by analyzing trivial shock-wave systems [3] with a single closing shock ($n = 0$) for extremum. It has been proved [3] that, for $f = p$ or $f = \rho$, the closing shock is normal ($\sigma = m$) and has intensity $J_m = (1 + \varepsilon)\mathrm{M}^2 - \varepsilon$, where $\varepsilon = (\gamma - 1)/(\gamma + 1)$. The maximum value of the velocity head ($f = \rho v^2 \equiv d$) is attained in a system with a weak ($\sigma = \alpha$) closing discontinuity ($J_\alpha = 1$).

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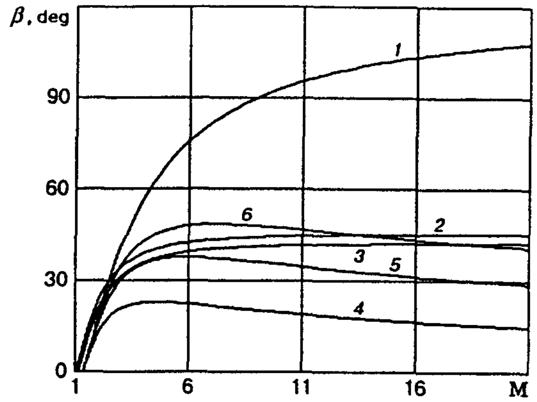

Fig. 1

In some technical devices, it is necessary to transform the supersonic flow to a subsonic flow with minimal losses of the total pressure, i.e., it is necessary to solve the problem

$$K_{n,\sigma}^{(p_0)} \to \max_{\mathrm{M}_{n,\sigma} \leqslant 1}, \tag{1.1}$$

where $\mathrm{M}_{n,\sigma}$ is the Mach number behind the system $S_{n,\sigma}$.

Beginning with [1, 2], systems that are optimal for $f = p_0$ have been considered with a normal closing shock ($S_{n,m}^{(p_0)}$).

Obviously, the condition $\mathrm{M}_{n,\sigma} \leqslant 1$ can also ensure an oblique shock with intensity in the range $[J_*(\mathrm{M}), J_m(\mathrm{M})]$, where the quantity $J_*(\mathrm{M})$ corresponds to the shock behind which the gas velocity is equal to the sound velocity [4]:

$$J_* = \frac{\mu - 1}{2\varepsilon} + \sqrt{\left(\frac{\mu - 1}{2\varepsilon}\right)^2 + \mu}, \qquad \mu = 1 + \varepsilon(\mathrm{M}^2 - 1). \tag{1.2}$$

In the system $S_{n,\sigma}$, the smaller the intensity of the closing shock, the smaller the losses of the total pressure. Therefore, to satisfy (1.1), it is reasonable to use a system $S_{n,*}$ in which the closing shock has intensity $J_\sigma = J_*$ (1.2).

The intensities of optimal waves chosen by analysis of the extrema of the variables f determine the flow deflection angles in the system β, and, hence, the optimal (for f) body geometry.

In this paper, we analyze the flow deflection angles in shock-wave systems that are optimal for the variables f for various types of closing shock. Emphasis is on flows with large Mach numbers.

2. The analysis in [3] of optimal shock-wave systems shows that maximum values of the functions studied are reached in systems that consist of a simple isentropic wave i and a closing shock σ.

The intensities $J_i^{(f)}$ of optimal isentropic waves in a flow with a given M value must ensure special Mach numbers M_f ahead of the closing shock: $\mathrm{M}_1 = \mathrm{M}_f$.

In the cases $f = d$ and $f = \rho$, we have $\mathrm{M}_d = \mathrm{M}_\rho = \sqrt{2}$, and the closing shock is a weak discontinuity ($J_\sigma = 1$) for $f = d$ and a normal shock for $f = \rho$. For the variable p, we have $\mathrm{M}_p = \sqrt{(2 - \varepsilon)/(1 - \varepsilon)}$ and $J_\sigma = J_m(\mathrm{M}_p)$, and for p_0, we have $\mathrm{M}_{p_0} = 1$ and $J_\sigma = 1$.

For an optimal isentropic wave, the flow deflection angle is determined from the formula

$$\beta_{s,\sigma} = \omega(\mathrm{M}) - \omega(\mathrm{M}_f), \tag{2.1}$$

where $\omega(\mathrm{M})$ is the Prandtl–Meyer function.

Since, for a weak discontinuity and a normal shock, the deflection angles are equal to zero, for systems $S_{1,\alpha}^{(d)}$ and $S_{1,m}^{(\rho)}$ that are optimal for the velocity head and density, the flow deflection angles are equal and are calculated from formula (2.1) for $\mathrm{M}_f = \sqrt{2}$ (curve 1 in Fig. 1; here and below, all calculations are performed

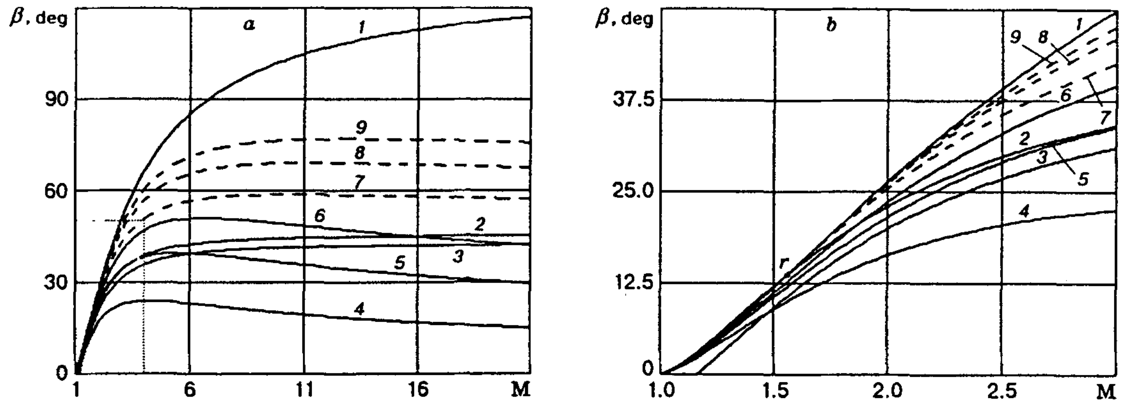

Fig. 2

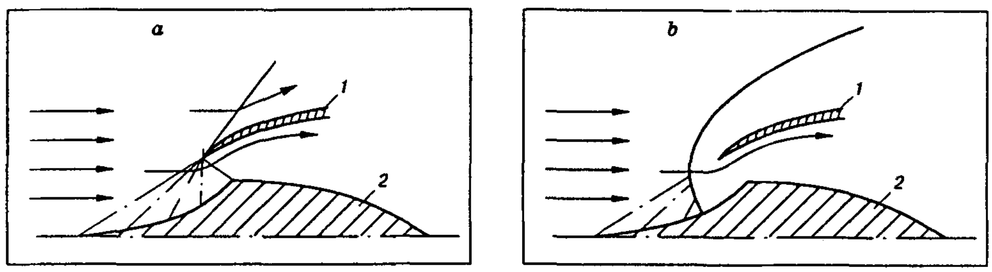

Fig. 3

for $\gamma = 1.4$).

In a system that is optimal for static pressure, the total flow deflection angle also coincides with the flow deflection angle for an isentropic wave and is determined from (2.1) for $\mathrm{M} = \mathrm{M}_p$ [the dependences $\beta(\mathrm{M})$ that correspond to systems optimal for static pressure differ only slightly from similar functions constructed for velocity head and density, and, therefore, they are not given separately].

In the case $f = p_0$, the optimal system is a compression wave behind which the flow velocity is equal to the sound velocity. Hence, in this system, the intensities of the closing sonic and normal shocks are equal to unity. The flow deflection angles in the systems $S_{1,*}^{(p_0)}$ and $S_{1,m}^{(p_0)}$ are also determined from (2.1) provided that $\mathrm{M}_{p_0} = 1$ (curve 1 in Fig. 2a; the fragment shown by the dotted curve in Fig. 2a is scaled up in Fig. 2b).

As can be seen from Figs. 1 and 2, the deflection angles increase monotonically from zero to the limiting ($\mathrm{M} \to \infty$) value

$$\beta_{\lim} = \frac{\pi}{2} \frac{1 - \sqrt{\varepsilon}}{\sqrt{\varepsilon}} - \omega(\mathrm{M}_f). \tag{2.2}$$

At large Mach numbers of the incoming flow, the flow deflection angles in the compression wave differ only slightly from the limiting value (2.2).

Beginning with a certain Mach number $\mathrm{M}_r^{(f)}$ (see Table 1), the angle $\beta_{s,\sigma}$ exceeds the limiting flow deflection angle β_l for an oblique shock. The latter is calculated from the relations [4]

$$J_l = \frac{\mu - (1+\varepsilon)}{2\varepsilon} + \sqrt{\left(\frac{\mu - (1+\varepsilon)}{2\varepsilon}\right)^2 + \frac{\mu(1+2\varepsilon) - 1}{\varepsilon}},$$

$$\beta_l = \arctan\left[\sqrt{\frac{J_l - 1}{J_l + \varepsilon}} \frac{(1+\varepsilon) + (J_l + \varepsilon)}{1 + \varepsilon J_l} \frac{(1-\varepsilon)(J_l - 1)}{2(J_l + \varepsilon)}\right].$$

This constrains the use of systems with isentropic waves in real technical objects. In particular, in

TABLE 1

Parameters	$M_r^{(p)}$	$M_r^{(d)}$	$M_r^{(p_0)}$	$M_\beta^{(d)}$	$M_\beta^{(p)}$	$M_g^{(1)}$	$M_g^{(2)}$	$M_g^{(3)}$	$M_g^{(4)}$
M	2.66	2.51	1.56	4.62	4.73	1.87	6.63	15.78	20.49
J_1	6.14	5.31	2.10	7.58	7.65	1.42	4.95	10.36	12.57

supersonic external-compression inlet diffusers, this constraint is associated with the presence of a shell [5] (Fig. 3). It is generally believed that the inner wall of the shell should be oriented streamwise behind the closing shock σ. If it is assumed that the flow deflection angle in the inlet duct of a plane diffuser (between the shell 1 and the central body 2) coincides with the slope of the outer surface of the shell [i.e., if it is assumed that the angle β_ω between the inner and outer surfaces of the shell tends to zero (Fig. 3a)], the constraint

$$\beta_{s,\sigma} < \beta_l(\mathrm{M}) \tag{2.3}$$

must be imposed on the total flow deflection angle $\beta_{s,\sigma}$ in the system.

In real diffusers, $\beta_\omega = 3$–$5°$, and, hence, the constraint on the angle $\beta_{s,\sigma}$ is more stringent:

$$\beta_{s,\sigma} < \beta_l(\mathrm{M}) - \beta_\omega. \tag{2.4}$$

If (2.4) is not satisfied, a detached curved shock is formed ahead of the shell (Fig. 3b), and this leads to deterioration of the gas-dynamic characteristics of the inlet diffuser.

The constraints (2.3) and (2.4) are shown in Figs. 1 and 2 (curves 2 and 3, respectively). These curves have points of intersection with curve 1 (point r in Fig. 2b). The values of $\mathrm{M}_r^{(f)}$ that correspond to these points are the upper limits of existence of optimal systems with isentropic waves.

Since the Mach numbers $\mathrm{M}_r^{(f)}$ are relatively small (see Table 1), it is technically difficult to organize optimal systems with isentropic waves for supersonic velocities of the undisturbed flow.

3. In systems $S_{n,\sigma}$ with oblique shocks, maximum values of the variables f are reached if the intensities of the first n shocks are equal ($J_1 = J_2 = \ldots = J_n \equiv J$) [2, 3]. The values of $J^{(d)}$ and $J^{(p)}$ that ensure maximum values of the coefficients $K_{n,\alpha}^{(d)}$ and $K_{n,m}^{(\rho)}$ are determined from the formula

$$\mu = \frac{J^{n-1}(1+\varepsilon J)^{n+1}}{(J+\varepsilon)^{n-1}(1+\varepsilon)}, \tag{3.1}$$

and the intensities $J^{(p)}$ that lead to a maximum of the function $K_{n,m}^{(p)}$ are determined from the formula

$$\mu = \frac{J^{n-1}(1+\varepsilon J)^{n+1}}{(J+\varepsilon)^{n-1}(1+\varepsilon)\,(1-\varepsilon^2)}. \tag{3.2}$$

In the system $S_{n,m}^{(p_0)}$, the total pressure recovery factor reaches an extremum at intensities $J^{(p_0)}$, which are found from the equation

$$\mu = A\frac{J^{n-1}(1+\varepsilon J)^n}{(J+\varepsilon)^n}[J^2 + 2BJ + 1 + (J-1)\sqrt{J^2+2CJ+1}], \tag{3.3}$$

where $A = \varepsilon(2+\varepsilon)/4(1+\varepsilon)^2$, $B = (\varepsilon^2+2\varepsilon+2)/\varepsilon(2+\varepsilon)$, and $C = \varepsilon(3\varepsilon+4)/(2+\varepsilon)^2$.

In the systems listed above, the intensity J_σ of the closing shock σ differs from $J^{(f)}$, and only in the optimal system $S_{n,*}^{(p_0)}$ are the intensities of all shocks, including the closing shock, equal and given explicitly by the formula

$$J_1 = J_2 = \ldots = J_{n+1} = \frac{\alpha-1}{2\varepsilon} + \sqrt{\left(\frac{\alpha-1}{2\varepsilon}\right)^2 + \alpha}, \qquad \alpha = \mu^{1/(1+n)}. \tag{3.4}$$

Knowing the Mach number of the incoming flow and having determined the wave intensities for the system from (3.1)–(3.4), it is not hard to calculate the flow deflection angles in the system optimal for the variable f.

4. The flow deflection angles in optimal systems consisting of one oblique shock and a closing shock are given in Figs. 1 and 2 (curves 4). In contrast to optimal systems with isentropic waves, the flow deflection angles in the systems considered behave nonmonotonically with variation in M and have a maximum for $M = M_\beta^{(f)}$. If $M > M_\beta^{(f)}$, the angles decrease with increase in M and tend to a constant value as $M \to \infty$.

Simple analysis shows that in the optimal systems $S_{1,\alpha}^{(d)}$ and $S_{1,m}^{(\rho)}$ (curve 4 in Fig. 1), the maximum value of the angle $\beta^{(d)} = \beta^{(\rho)}$ is reached for the oblique shock intensity

$$J_\beta^{(d)} = J_\beta^{(\rho)} = \frac{\sqrt{\varepsilon} + (1+\varepsilon) + (1+\sqrt{\varepsilon})\sqrt{1+\varepsilon}}{\sqrt{\varepsilon}},$$

and, for the optimal system $S_{1,m}^{(p)}$, the maximum flow deflection angle corresponds to

$$J_\beta^{(p)} = \frac{\sqrt{\varepsilon}(4+\varepsilon-\varepsilon^2) + (1+\varepsilon)\sqrt{(4-\varepsilon^2)(4-\varepsilon)}}{2\sqrt{\varepsilon}}.$$

The values of $M_\beta^{(d)}$ and $M_\beta^{(p)}$ are given in Table 1.

As $M \to \infty$, the flow deflection angles in the optimal systems $S_{1,\alpha}^{(d)}$, $S_{1,m}^{(\rho)}$, and $S_{1,m}^{(p)}$ tend to zero. This makes it possible to satisfy conditions (2.3) and (2.4) for any Mach numbers from the interval $[M_f^{(f)}, \infty)$.

Noteworthy is the difference in behavior between the deflection angles in the systems $S_{1,m}^{(p_0)}$ and $S_{1,*}^{(p_0)}$, which are optimal for the total pressure.

As can be seen from Fig. 2 (curve 4), the character of the function $\beta_{s,m}(M)$ in the system $S_{1,m}^{(p_0)}$ does not differ qualitatively from the behavior of this function in the systems $S_{1,\alpha}^{(d)}$, $S_{1,m}^{(\rho)}$, and $S_{1,m}^{(p)}$. In particular, $\beta_{s,m} \to 0$ as $M \to \infty$, and this makes it possible to satisfy conditions (2.3) and (2.4) for any Mach numbers.

In the system $S_{1,*}^{(p_0)}$, the character of the function $\beta_{s,*}(M)$ is different. It can be seen from Fig. 2a and b (curve 7) that condition (2.3) for the function $\beta_{s,*}(M)$ is satisfied only for small Mach numbers, and condition (2.4) is not satisfied at all. This is due to the fact that in the system $S_{1,*}^{(p_0)}$, only the deflection angle β_1 at the first shock behaves nonmonotonically; it reaches a maximum at a certain M and tends to zero as $M \to \infty$. In contrast to β_1, the flow deflection angle at the closing shock increases monotonically with increase in M and tends to the limiting value [2]

$$\beta_h = \arctan\frac{1-\varepsilon}{2\sqrt{\varepsilon}}, \tag{4.1}$$

where $\beta_h = 45.58°$.

With increase in M, the total angle $\beta_{1,*}$ increases, reaches a maximum [($\beta_{\max} > \beta_h$ (4.1)], and then approaches the angle β_h from above (curve 7 in Fig. 2a).

Note that in the region bounded by inequality (2.3), the flow deflection angle in an optimal system with an isentropic wave is smaller than the flow deflection angle in an optimal shock-wave system. Hence, the shock-wave system $S_{1,*}^{(p_0)}$ is ineffective for both large values of M [compared with the system $S_{1,m}^{(p_0)}$ from the viewpoint of conditions (2.3) and (2.4)] and small values (compared with the system $S_{1,*}^{(p_0)}$ for an isentropic wave from the viewpoint of the total pressure recovery factor).

However, in technical devices for which inequalities (2.3) and (2.4) are not essential, with large Mach numbers, it is reasonable to use the system $S_{1,*}^{(p_0)}$ instead of the system $S_{1,m}^{(p_0)}$.

5. An increase in the number of shocks ($n > 1$) in the optimal systems $S_{n,\sigma}^{(f)}$ does not change qualitatively the dependence of M on the total flow deflection angles (see curves 5 and 6 in Figs. 1 and 2, which correspond to the systems $S_{n,\alpha}^{(d)}$, $S_{n,m}^{(\rho)}$, and $S_{n,m}^{(p_0)}$, and also curves 8 and 9, which correspond to the system $S_{n,*}^{(p_0)}$, constructed for $n = 2$ and 3, respectively). As n increases, so does $\beta_{\max}$, and the maximum is shifted toward larger Mach numbers. As with $n = 1$, the angle $\beta_{s,\sigma} \to 0$ as $M \to \infty$ in all systems with $n > 1$, with the exception of $S_{n,*}^{(p_0)}$, for which it tends to the limiting deflection angle (4.1).

There is, however, a fundamental difference in the behavior of curves for $n = 1$ and $n > 1$. This difference is that, for systems with $n > 1$, curves 5 and 6 can intersect with curves 2 and 3, which correspond to conditions (2.3) and (2.4).

Thus, for example, in Fig. 2a and b, curve 6, which corresponds to the flow deflection angle in a system that is optimal for p_0 and consists of three oblique shocks and a closing normal shock, lies below curve 2 in the range $\mathrm{M} \in [1, \mathrm{M}_g^{(1)}]$. At the point with $\mathrm{M} = \mathrm{M}_g^{(1)}$ (see Table 1), these curves intersect, and, for $\mathrm{M} > \mathrm{M}_g^{(1)}$, the total flow deflection angle in the system $S_{3,m}^{(p_0)}$ turns out to be larger than $\beta_l(\mathrm{M})$ (Fig. 2b). As a result, for $\mathrm{M} > \mathrm{M}_g^{(1)}$, conditions (2.3) and (2.4) are not satisfied.

For $\mathrm{M} = \mathrm{M}_g^{(2)}$, the total deflection angle reaches a maximum, then decreases, and tends to zero as $\mathrm{M} \to \infty$. For $\mathrm{M} = \mathrm{M}_g^{(3)}$, curve 6 intersects curve 2 again, and, at the point $\mathrm{M} = \mathrm{M}_g^{(4)}$, it intersects curve 3 (Fig. 2a). Hence, as with small M, for any values of M from the range $[\mathrm{M}_g^{(4)}, \infty)$, an optimal multishock system can be realized which is used for the recovery of the total pressure in supersonic inlet diffusers.

This is also true for other values of n.

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